A Reducibility of the Kampé de Fériet Function

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Abstract: In 1928, Bailey has obtained the very interesting result known as Kummer’s second theorem by a method entirely different from Kummer’s method. The aim of this research not is to obtain one interesting result of reducibility of Kampé de Fériet function using the Gauss’s second summation theorem on the sum of a \( 2F_1 \) by following the same technique developed by Bailey.

Key Words: Kampé de Fériet function; Gauss’s Second summation theorem.

1 Introduction

We recall the definition of generalized Kampé de Fériet’s function as follows [5]

\[
\sum_{r,s=0} \frac{\prod_{j=1}^{p+q} (a_j)_r \cdot \prod_{j=1}^{m} (b_j)_s \cdot x^r y^s}{\prod_{j=1}^{q} (c_j)_r \cdot \prod_{j=1}^{n} (\gamma_j)_s \cdot r! s!}.
\]

(1.1)

Where for convergence

(i) \( p + q < \ell + m + 1, p + k < \ell + n + 1, |x| < \infty, |y| < \infty \) or

(ii) \( p + q < \ell + m + 1, p + k < \ell + n + 1 \) and

\[
\left\{ \begin{array}{l}
|x|^{\frac{1}{\ell - 1}} + |y|^{\frac{1}{\ell - 1}} < 1, \text{if } p > \ell \\
\max \{|x|, |y|\} < 1, \text{if } p \leq \ell
\end{array} \right.
\]

Although the double hypergeometric series defined by (1.1) reduces to the Kampé de Fériet function in the special case:

\( q = k \) and \( m = n \)

yet it is usually referred to in the literature as the Kampé de Fériet series.

The following are the cases in which the Kampé de Fériet function defined in (1.1) can be expressed in terms of generalized hypergeometric series.

\[
\sum_{r,s=0} \frac{\prod_{j=1}^{p+q} (a_j)_r \cdot \prod_{j=1}^{m} (b_j)_s \cdot x^r y^s}{\prod_{j=1}^{q} (c_j)_r \cdot \prod_{j=1}^{n} (\gamma_j)_s \cdot r! s!}.
\]

(1.1)

and

\[
\sum_{r,s=0} \frac{\prod_{j=1}^{p+q} (a_j)_r \cdot \prod_{j=1}^{m} (b_j)_s \cdot x^r y^s}{\prod_{j=1}^{q} (c_j)_r \cdot \prod_{j=1}^{n} (\gamma_j)_s \cdot r! s!}.
\]

(1.1)

where, and in what follows, \( \Delta (\ell ; \lambda) \) abbreviates the array of \( \ell \) parameters

\[
\frac{\lambda}{\ell} \begin{pmatrix} \lambda + 1 & \cdots & \lambda + \ell - 1 \\ \ell & \cdots & \ell \end{pmatrix}, \quad \ell = 1, 2, 3, ...
\]

For more detail see [5, pp. 28-32]. In 1928, Bailey [2] has obtained the following very interesting result known as Kummer’s second theorem, viz.

\[
e^{-\frac{x}{2}} F_1 (\alpha; 2\alpha; -x) = 0 F_1 (-\alpha + \frac{1}{2}; \frac{x^2}{16})
\]

(1.6)
by a method entirely different from Kummer’s method. Bailey has obtained the result with the help of Gauss’s
second summation theorem on the sum of \( \text{a}_2 \text{F}_1 \), viz.

\[
\text{a}_2 \text{F}_1 \left[ \frac{1}{2} \left( \frac{1}{2} + a + b \right) \left| \frac{1}{2} \right. \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b + \frac{3}{2} \right)}
\]

provided \( a + b + 1 \neq 0, -2, -4, -6, \ldots \). (1.7)

In 1992, Lavoie et al. [4] have obtained a large number of summation formulae closely related to (1.7) of which one is given below

\[
\text{a}_2 \text{F}_1 \left[ \frac{1}{2} \left( \frac{1}{2} + a + b \right) \left| \frac{1}{2} \right. \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} a + \frac{1}{2} b + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} b + \frac{3}{2} \right)} - \frac{2}{\Gamma \left( \frac{1}{2} a \right) \Gamma \left( \frac{1}{2} b \right)}
\]

(1.8)

If we put \( a = a + 2n \) and \( b = -2n \), where \( n \) is zero or a positive integer, then we get

\[
\text{a}_2 \text{F}_1 \left[ \frac{a + 2n, -2n}{\left( \frac{1}{2} a + \frac{3}{2} \right) \left| \frac{1}{2} \right. \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a + 2n - \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} a + 2n + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a + n + \frac{1}{2} \right) \Gamma \left( -n + \frac{1}{2} \right)}
\]

(1.9)

On the other hand if we put \( a = a + 2n + 1 \) and \( b = -2n - 1 \), where \( n \) is zero or a positive integer, then we get

\[
\text{a}_2 \text{F}_1 \left[ \frac{a + 2n + 1, -2n - 1}{\left( \frac{1}{2} a + \frac{3}{2} \right) \left| \frac{1}{2} \right. \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a + 2n + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} a - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} a + 2n + \frac{3}{2} \right) \Gamma \left( \frac{1}{2} a + n + \frac{1}{2} \right) \Gamma \left( -n + \frac{1}{2} \right)} - \frac{2}{\Gamma \left( \frac{1}{2} a \right) \Gamma \left( \frac{1}{2} b \right)}
\]

(1.10)

The aim of this research note is to obtain one result closely related to (1.6) by employing the summation formulae (1.9 and 1.10).

2. Results Required

The following results will be required in our present investigations.

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} A(n, m - n) \quad (2.1)
\]

\[
(\alpha)_{m-n} = \frac{(-1)^n \Gamma(\alpha + m)}{\Gamma(\alpha)(1 - \alpha - m)_n} \quad (2.2)
\]

\[
(m - n)! = \frac{(-1)^n (m)!}{(-m)_n} \quad (2.3)
\]

\[
(\alpha)_{2m} = 2^{2m} \left( \frac{1}{2} \alpha \right)_m \left( \frac{1}{2} \alpha + \frac{1}{2} \right)_m \quad (2.4)
\]

\[
(\alpha)(\alpha - 2m) = \frac{(-1)^{2m} \Gamma(\alpha)}{(1 - \alpha)_{2m}} \quad (2.5)
\]

\[
(\alpha - m) = \frac{(-1)^m \Gamma(\alpha)}{(1 - \alpha)_m} \quad (2.6)
\]

\[
(2m)! = 2^{2m} \left( \frac{1}{2} \right)_m (m)! \quad (2.7)
\]

\[
(2m + 1)! = 2^{2m} \left( \frac{3}{2} \right)_m (m)! \quad (2.8)
\]

\[
(\alpha)_{2m+1} = \alpha 2^{2m} \left( \frac{1}{2} \alpha + \frac{1}{2} \right)_m \left( \frac{1}{2} \alpha + 1 \right)_m \quad (2.9)
\]

3. Main Result

The following result for reducibility of Kampé de Fériet Function will be established in this section.

\[
\Gamma_{u,1}^{\lambda,1} \left[ \frac{\alpha}{\tau, \beta}, \frac{1}{2}, \alpha + 2 \right| \frac{1}{2} - \frac{1}{x}, \frac{1}{x} \right] = 2^{\lambda+1} F_{2\nu+2}
\]
4. Derivation

To prove (3.1), we proceed as follows:

Let \( I = \sum_{m=0}^{\infty} A(m) \) (Let)

Now, using series identity

\[ \sum_{m=0}^{\infty} A(m) = \sum_{m=0}^{\infty} A(2m) + \sum_{m=0}^{\infty} A(2m+1) \]

We have,

\[ \sum_{m=0}^{\infty} A(2m) = \frac{\left(\alpha_{\lambda}\right)_{2m} \left(\alpha\right)_{2m} x^{2m}}{\left(\tau_{v}\right)_{2m} (2\alpha + 2)_{2m}} \]

using (2.4), (2.5), (2.6), and (2.7), we get

\[ \sum_{m=0}^{\infty} A(2m) = \frac{\left(\alpha_{\lambda}\right)_{2m} \left(\alpha\right)_{2m} x^{2m}}{\left(\tau_{v}\right)_{2m} (2\alpha + 2)_{2m}} \frac{\Gamma(\frac{1}{2})}{\Gamma(1-\alpha-2m) \Gamma(-\alpha-1) \Gamma(-1-\alpha-2m)} \]

using (2.2) and (2.3), we get

\[ \sum_{m=0}^{\infty} A(2m) = \frac{\left(\alpha_{\lambda}\right)_{2m} \left(\alpha\right)_{2m} x^{2m}}{\left(\tau_{v}\right)_{2m} (2\alpha + 2)_{2m}} (1-\alpha-2m) \frac{\Gamma(\frac{1}{2})}{\Gamma(-\alpha-m) \Gamma(-m+\frac{1}{2})} \]

Also,

\[ \sum_{m=0}^{\infty} A(2m+1) = \sum_{m=0}^{\infty} A(2m+1) \]

\[ \sum_{m=0}^{\infty} A(2m+1) = \frac{\left(\alpha_{\lambda}\right)_{2m+1} \left(\alpha\right)_{2m+1} x^{2m+1}}{\left(\tau_{v}\right)_{2m+1} (2\alpha + 2)_{2m+1} (2m+1)!} \]
using (1.10), we get
\[
\sum_{m=0}^{\infty} A(2m+1) =
\left(\frac{\alpha_+}{\tau_+}\right)_{2m+1} \left(\frac{\alpha}{\tau_+}\right)_{2m+1} x^{2m+1}
\]

\[
\frac{(\alpha_+)^{2m+1}}{(\tau_+)^{2m+1} (2\alpha + 2m + 1) !}
\]

\[
= -2 \frac{\Gamma(1/2) \Gamma(-\alpha - 2m) \Gamma(-1 - \alpha)}{\Gamma(1 - \alpha) \Gamma(1 - \alpha - m) \Gamma(-m - \frac{1}{2})}
\]

\[
\sum_{m=0}^{\infty} A(2m+1) =
\frac{-\alpha_+ x 2^{2(\lambda+\mu+\alpha)} m (\alpha_+ + 1)_m (\alpha_+ + 1)_m x^{2m}}{(2\alpha + 2) \tau_+ (\alpha_+ + 1)_m (\alpha_+ + 1)_m (\alpha + 3)_m m!}
\]

\[
\sum_{m=0}^{\infty} A(2m+1) =
\frac{-x \alpha_+}{2\tau_+ (\alpha_+ + 1)} 2\lambda x F_{2\lambda + 1}
\]

\[
\left[
\begin{array}{c}
\frac{1}{2} (\alpha_+ + 1), \frac{1}{2} (\alpha_+ + 2)
\end{array}
\right]
\]

\[
\left[
\begin{array}{c}
\frac{1}{2} (\tau_+ + 1), \frac{1}{2} (\tau_+ + 2), \frac{1}{2} (2\alpha + 3)
\end{array}
\right]
\]

\[
\left[
\begin{array}{c}
\alpha_+ + 1
\end{array}
\right]
\]

\[
\frac{2^{2(\lambda+\mu+\alpha)} x^2}{16}
\]

(4.3)

Substituting the values from (4.2) and (4.3) in (4.1), we get the desired result (3.1).

5. Special Cases

If we take \( \lambda = \mu = 0 \) in (3.1), we get

\[
e^{-\frac{x}{2}} \left[ \begin{array}{c} \alpha; 2\alpha + 2; x \end{array} \right] =
\left[ \begin{array}{c} \frac{1}{2} (\alpha + 3); \frac{1}{2} (\alpha + 1), \frac{1}{2} (2\alpha + 3) \end{array} \right] \frac{x^2}{16}
\]

\[
- \frac{x}{2(\alpha + 1)} \left[ \begin{array}{c} \alpha_+; 2\alpha_+ + 3; x \end{array} \right] \frac{x^2}{16}
\]

(5.1)

This result has been obtained by Kim et al. [3].

References